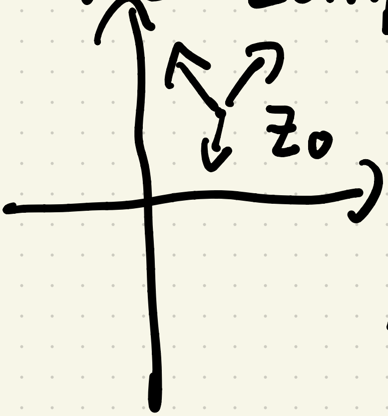
29/09 Math 2230A

Complex Variables with Applications

1. Differentiability

Def: $\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$ exists.

Remark: Any direction on the complex plane.



For a multi-variable, it may not has this property, e.g. $f = x$. (From x-axis, it is 1. From y-axis, it is 0.)

2. Cauchy Riemann Equation (Abbreviated as C-R).

Intuition (Necessary Condition)

For a differentiable function. Δz can have any direction. Then we take the direction to be from real axis and imaginery axis,

$$\partial_x f = \partial_y f \Rightarrow$$

$$u_x + v_x i = \frac{\partial u}{\partial y}\frac{\partial y}{\partial iy} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial iy} i$$

$$= -i u_y + v_y$$

$$\Rightarrow u_x = v_y, \quad u_y = -v_x.$$

Sufficiency:

Need one more condition:

f is continnously differentiable (C^1) w.r.t $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$.

Typical Counter Example when no C^1 assumption:

$$f = \begin{cases} \bar{z}^2/z, & z \in \mathbb{C}/\{0\} \\ 0, & z = 0 \end{cases}.$$

Proof: $f(z_0+\Delta z) - f(z_0)$

$$= U_x(z_0)\Delta x + U_y(z_0)\Delta y + i(V_x(z_0)\Delta x + V_y(z_0)\Delta y) + \varepsilon\Delta x + \varepsilon\Delta y.$$

Divide Δz from both sides, and notice that $\left|\frac{\Delta x}{\Delta z}\right|, \left|\frac{\Delta y}{\Delta z}\right| \leq 1$. Then take Δz goes to 0, and plug C-R in, we achieve

$$f'(z_0) = u_x + v_x i.$$

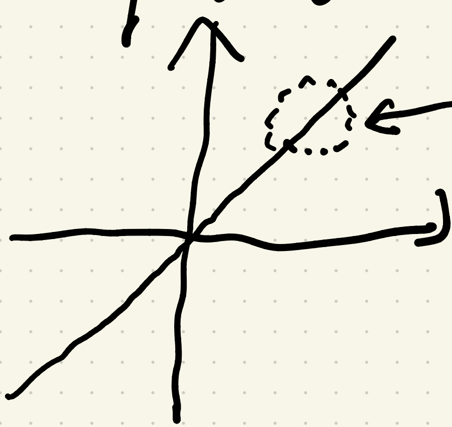
3. Analicity

Require f to be differentiable on an open set.

Counter Exp for differentiable but not analytic function:

$$f(z) = x^2 + y^2 i$$

$$\xRightarrow{C-R} \begin{cases} 2x = 2y \\ 0 = 0 \end{cases} \Rightarrow \begin{cases} x = y \\ 0 = 0 \end{cases}.$$



Cannot be differentiable on open sets.

3 Identities with Trigonometric and Hyperbolic Function

$$\sinh z = \frac{e^{z} - e^{-z}}{2} = \frac{e^{x}e^{yi} - e^{-x}e^{-yi}}{2}$$

$$= (e^{x}\cos y + e^{x}\sin y\, i - e^{-x}\cos y + e^{-x}\sin y\, i)/2$$

$$= \frac{e^{x} - e^{-x}}{2}\cos y + \frac{e^{x} + e^{-x}}{2}\sin y\, i$$

$$= \sinh x \cos y + \cosh x \sin y\, i$$

$$|\sinh x|^2 = \frac{e^{2x} - 2 + e^{-2x}}{4}\cos^2 y + \frac{e^{2x} + 2 + e^{-2x}}{4}\sin^2 y$$

$$= \frac{e^{2x}}{4} + \frac{e^{-2x}}{4} - \frac{1}{2}(\cos^2 y - \sin^2 y)$$

$$= \frac{e^{2x} - 2 + e^{-2x}}{4} + \frac{1}{2}(1 - \cos^2 y + \sin^2 y)$$

$$= \sinh^2 x + \sin^2 y$$

Some of assignments needing attention.

P35 - 6. (Rigorous Proof)

Suppose we have a continuous path $\gamma: [0,1] \to B(0,1) \cup B(2,1)$. connecting $z_1 \in B(0,1)$, $z_2 \in B(2,1)$.

Take $\tau = \inf\{t \mid \gamma(t) \in B(2,1)\}$.

Then by continuity, τ has a neighborhood s.t its image contains in $B(0,1)$ or $B(2,1)$. But its impossible due to the def for τ. (infimum).

P103-6.

This question itself is of no interest, but principal value of $|z|^a$ is very important.

If not, $|z|^a = \exp(a \log |z|)$

$= \exp(a \ln|z| + 2an\pi i)$

$= |z|^a e^{2an\pi i}$.

If $a \in \mathbb{Z}$, it is fine.

If a is fractional, say $a = \frac{1}{2}$ then it will be $|z|^a$ for even n and $-|z|^a$ for odd n.

Additional:

Conformality for Analytic functions.

For a path $\gamma : [0,1] \to \mathbb{C}$, assuming enough regularity.

Then $(f(\gamma(t)))' = f'(\gamma(t))\gamma'(t)$

$\Rightarrow \arg(f'(\gamma(t))\gamma'(t)) = \arg(f'(\gamma(t)))$

$+ \arg(\gamma'(t))$, if $f'(t)$ & $\gamma'(t)$ are nonzero. (It makes no sense to talk angle for 0!)

Now we may have 2 paths, γ_1 & γ_2. By the above computation

$\arg((f(\gamma_1(t)))') - \arg((f(\gamma_2(t)))')$

$= \gamma_1'(t) - \gamma_2'(t)$. $(\gamma_1(t) = \gamma_2(t))$.

This shows that angle between 2 paths at $f'(z_0) \neq 0$, is preserved after the analytic map f, of course $\gamma_1'(t)$ and $\gamma_2'(t)$ are non-zero.